An Extension of the Penalty Function Formulation to Incompressible Hyperelastic Solids Described by General Measure of Strain

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SYNOPSIS

The penalty function formulation for incompressible hyperelastic solids was first proposed about 30 years ago. Since then all studies have been limited to invariant type formulation of the strain energy function, although it is well known that this formulation does not correctly describe the behavior of a real material. On the other hand more realistic constitutive equations, based on general measures of the strain only, have been incorporated to mixed finite element algorithms. In this article, a penalty function formulation is proposed for the analysis of stress field in materials with constitutive equations based on the general measure of strain. The reduced integration method is used to weaken the penalty constraint in order to obtain meaningful numerical results. The incremental equilibrium equations are solved using the regular Newton–Raphson algorithm. The method is applied to evaluate the stress field in materials subjected to plane strain conditions. Satisfactory agreements have been obtained with analytical solutions when available. © 1995 John Wiley & Sons, Inc.

INTRODUCTION

The stress analysis of rubber-like solids offers two unique features as compared to the analysis of more conventional structural materials; first, these materials are nearly incompressible and second, they are capable of experiencing large deformations. For linear incompressible solids two methods of analysis have been employed: the mixed finite element method (MFEM)[1,2] and the penalty finite element method (PFEM).[3-7] The first method of analysis treats the displacements and the hydrostatic pressure degrees of freedom as the unknown variables. The second method treats the material as nearly incompressible and uses FE models in which the displacement degrees of freedom are the only unknowns in the system of the discretized FE equations.

Similar techniques have been employed for the stress analysis of nonlinear rubber-like solids. The mixed FE technique has been successfully employed for the numerical solution of hyperelastic materials[8-13] and the PFEM has been extensively discussed in the following references.[14-16] Recently, the so-called augmented Lagrangian method was employed for the numerical solution of the rubber-like solids.[17] Compared to the MFEM the PFEM has fewer independent variables and hence potentially requires less computer time.

All the proposed PFE models for rubber-like materials assumed that the constitutive equations are derived from the invariant type formulation of the strain energy function.[18] However, it is well known that this formulation does not yield the appropriate constitutive equations for the real description of the response of hyperelastic solids. In the last 30 years more realistic strain energy functions methods were published in order to describe the correct response of these materials. The most of the available strain energy functions are based on the Valanis–Landel hypothesis.[19] For instance, we might mention the strain energy functions developed by Ogden[20,21] and Crossland and Van der Hoff.[22] Also the Blatz et al.[23] strain energy function predicts the response of a hyperelastic solids but does not follow the above mentioned hypothesis.

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The scope of this article is to develop a penalty function FE algorithm based on the principal axes formulation. The Ogden's strain energy function was employed for the development of the appropriate constitutive equations. An alternative method of analysis based on the mixed FE technique was developed by Duffett et al.[24,25]

The fundamental kinematic concepts that are needed along the course of this study are presented and the fundamental concepts of the general measure of strain are discussed. These concepts have been applied by Ogden[20] for the description of the elastic large deformation of the rubber-like solids. The Ogden's strain energy function is briefly examined. The first order elastic moduli for hyperelastic solids, based on the Ogden's strain energy function are discussed, and the *penalty function* approach is applied in order to eliminate the pressure degree of freedom when the material is subjected to *plane strain* conditions. We also develop the material properties matrix for plane strain conditions. The general principle of virtual work[26] for elastic solids is discussed. The total Lagrangian formulation[27] was adopted for the linearization of the differential form of the equilibrium equations. The discretization of the equilibrium equations using the FE technique is discussed. The stiffness matrix was developed for plane strain conditions. In the case of plane strain conditions the reduced integration technique[27-30] was applied for the numerical evaluation of the penalty terms.

Two numerical examples were run according to the proposed model in order to confirm the validity of this algorithm. The first example deals with the uniform tension of a rubber sheet with a small circular hole located at the center of the sample; the second example deals with an infinitely long thick-walled cylinder subjected to internal pressure.

FUNDAMENTAL KINEMATICS

Consider a material point P of a solid body B. Suppose that P occupies the position $\underline{X}$ when B is in a reference configuration at $t = 0$. Let P occupy the instantaneous position $\underline{x}$ at time t. Then the motion of P may be described by

$$\underline{x} = \xi(\underline{X}, t). \tag{1}$$

Thus the instantaneous location $\underline{x}$ may be represented as a function of the undeformed location $\underline{X}$ and the time t. At an increment of time Δt the material point P is located at $\underline{x}^*$. The equation of motion from the undeformed state to the deformed configuration at $t = t + \Delta t$ is given by

$$\underline{x}^* = \xi(\underline{X}, t + \Delta t). \tag{2}$$

The gradient $\mathbf{F}$ of $\underline{x}$ with respect to $\underline{X}$ is given by:

$$\mathbf{F} = \mathrm{Grad}(\underset{\sim}{\xi}) \tag{3}$$

where Grad is the gradient operator with respect to the $\underline{X}$. The $\mathbf{F}$ is called the *deformation gradient* tensor at the material point P at time t by the theorem of polar decomposition.[31,32] $\mathbf{F}$ may be split into a symmetric positive definite stretch tensor and an orthogonal rotation tensor. Thus

$$\mathbf{F} = \mathbf{RU} \tag{4}$$

where $\mathbf{R}$ is the rotation tensor and

$$\mathbf{RU} = \mathbf{VR} \tag{5}$$

where $\mathbf{U}$ and $\mathbf{V}$ are the right and left stretch tensors, respectively. In general, the calculation of the components of $\mathbf{U}$ and $\mathbf{V}$ is involved and it is preferable to introduce the deformation tensors

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^t\mathbf{F} \tag{6}$$

$$\mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^t \tag{7}$$

which are called the right and left Cauchy Green tensors, respectively. It follows from their definitions that both tensors are symmetric and positive definite.

GENERALIZED STRAIN MEASURE

A material body subjected to a deformation assumes a new configuration. It is convenient to define a measure of the deviation of this configuration from a suitably chosen reference configuration. Any measure will serve that determines the directions of the principal axes of the deformation and the magnitude of the deformation in these directions. Such a measure is called *strain.* We recognize it as being a measure of the difference in distance between two material particles in different configurations. A deformation of a body is most easily described by a body coordinate system.[31,32] It is convenient to adopt one system (the *Lagrangian* system) for the undeformed configuration and another (the *Eulerian* system) for the deformed configuration. The deformation can then be viewed as a transformation from one system to the other. Which one is chosen as the reference system is immaterial. Along the course of this study we adopt the Lagrangian system as the reference system. In the theory of finite elasticity the most widely used strain measures are the Green's strain tensor

$$\mathbf{E} = \left(\frac{\mathbf{C} - 1}{2}\right), \tag{8}$$

defined in terms of the Lagrangian coordinates, and the Almansi's strain tensor

$$\mathbf{e} = \left(\frac{\mathbf{I} - \mathbf{c}}{2}\right), \tag{9}$$

defined in terms of the deformed or Eulerian coordinates. The tensor **C** is the inverse of Finger's tensor, **B**. The matrices **C** and **B** are *symmetric* and can be diagonalized by orthogonal transformations, that is,

$$\mathbf{C} = \underline{N}\Lambda^2\underline{N}^{-1} \tag{10}$$

and

$$\mathbf{B} = \underline{n}\Lambda^2\underline{n}^{-1} \tag{11}$$

where **C** and **B** are defined via Eqs. (6) and (7). The matrices $\underline{\mathrm{N}}$ and $\underline{\mathrm{n}}$ are formed from the eigenvectors of the **C** and **B**, and Λ^2 defines the matrix of the eigenvalues of **C** equal with those of **B**. The matrix Λ is diagonal whose elements λ_1, λ_2, and λ_3 are the principal stretches. It can easily be proven[31] that the eigenvectors $\underline{\mathrm{n}}$ and $\underline{\mathrm{N}}$ are connected by:

$$\underline{n} = R\underline{N} \tag{12}$$

where $\underline{\mathrm{n}} = (n_1, n_2, n_3)$ and $\underline{\mathrm{N}} = (N_1, N_2, N_3)$.

We define a generalized Lagrangian strain tensor as follows:

$$\mathbf{E}_{\mathrm{lg}} = \underline{N}\Phi\underline{N}^{-1} \tag{13}$$

where Φ is a matrix function of Λ, that is, $\Phi = \Phi(\Lambda)$ and it satisfies the condition

$$\Phi(I) = 0. \tag{14}$$

A particular form of the generalized strain measure can be obtained by letting

$$\Phi(\Lambda) = \frac{\Lambda^\alpha - I}{\alpha} \tag{15}$$

where α is a strain parameter that can be any real number. The strain function defined by (15) is called the *α-measure of strain.* Substitution of (15) into (13) yields:

$$E_{I\alpha} = \underline{N}\left(\frac{\Lambda^\alpha - I}{\alpha}\right)\underline{N}^{-1}. \tag{16}$$

Obviously, for $\alpha = 2$ the Lagrangian strain tensor is reduced to E. The α-measure of strain was introduced by Seth[33] and has been applied by Ogden[20] and independently by Blatz et al.[23] for the description of the elastic large deformation behavior of rubber-like solids. The generalized strain measure has recently been applied to nonhomogeneous deformations in rubber-like solids.[34] The strain parameter α is a material parameter and must be determined by experiment.

ODGEN'S STRAIN ENERGY FUNCTION

Since the 1940s a large number of strain energy functions have been proposed for hyperelastic incompressible solids. Between them we might mention the Neo-Hookean and Mooney–Rivlin[17] strain energy functions based on the principal invariants of the right Cauchy–Green strain tensor. Unfortunately, these functions can predict the experimental data only up to the extension ratio 1.5. In 1972 Ogden[20,21] proposed a new form of the strain energy function based on the general measure of strain. The proposed function has the form:

$$W = \sum_{k=1}^{m} \sum_{j=1}^{3} \mu_k e_j^{(\alpha_k)} \tag{17}$$

where the α-measure of strain is given by

$$e_j^{(\alpha)} = \frac{\lambda_j^\alpha - 1}{\alpha} \tag{18}$$

with λ_j the principal stretch ratio.

The upper limit of the first summation m defines the number of terms needed in order to successfully describe the observed experimental stress–deformation relationship in simple tension, pure shear, and equibiaxial shear. The material parameters (μ_k, α_k) must be determined by a nonlinear least square fitting.

FIRST ORDER ELASTIC MODULI FOR HYPERELASTIC INCOMPRESSIBLE SOLIDS

For hyperelastic incompressible solids the eigenvalues of the second Piola–Kirchhoff stress tensor are given via the following equation[21]

$$T_j = \hat{T}_j - \frac{p}{\lambda_j^2}; \quad (j = 1\text{–}3) \tag{19}$$

where T_j is given by:

$$\hat{T}_j = \frac{\partial W}{\partial E_j} \tag{20}$$

where W defines the strain energy function given in (17), and E_j are the principal values of the Green stain tensor as defined by Eq. (9). The hydrostatic pressure p is determined from the boundary value problem. Substitution of (17) into (20) yields the proper number of the T_{KL}-tensor,

$$\hat{T}_j = \sum_{k=1}^{m} \mu_k \lambda_j^{\alpha_k - 2}. \tag{21}$$

The principal stretches λ_j^2 can be determined as a function of the components of the right Cauchy–Green strain tensor $\mathbf{C_{KL}}$. Indeed, the *characteristic equation* of the tensor $\mathbf{C_{KL}}$ is:

$$\det(C_{\mathrm{KL}} - \lambda^2 \delta_{\mathrm{KL}}) = 0. \tag{22}$$

For plane problems the solution of the above equation yields the principal stretches λ_j^2 as a function of the components of $\mathbf{C_{KL}}$ tensor. That is,

$$\lambda_{1,2}^2 = \left(\frac{C_{11} + C_{22}}{2}\right) \pm \tfrac{1}{2}\sqrt{(C_{11} + C_{22})^2 + 4C_{12}^2} \tag{23}$$

$$\lambda_3^2 = C_{33}. \tag{24}$$

The eigenvectors $\underline{\mathrm{N}}$ may readily be computed by the following equation:

$$(\underline{C} - \lambda^2 \underline{1})\underline{N} = \underline{0}. \tag{25}$$

For *isotropic* solids, the principal directions $\underline{\mathrm{N}}$ of the second Piola–Kirchhoff stress tensor and those of the right Cauchy–Green strain tensor $\mathbf{C}$ are the same. It may easily be seen from Eq. (25) that for two-dimensional problems the principal axes of the $\mathbf{C}_{ij}$ tensor are:

$$N_K = \left(\frac{C_{12}}{\Delta_K}\right) i_1 + \frac{(\lambda_K^2 - C_{11})}{\Delta_K} i_2; \quad K = 1, 2 \tag{26}$$

and

$$N_3 = i_3. \tag{27}$$

The vectors i_k (k = 1–3) define an orthonormal Cartesian basis with respect to the reference Lagrangian coordinate system. The functions Δ_K are given by $\Delta_K = ((\lambda_K^2 - C_{\mathrm{KK}})^2 + C_{12}^2)^{1/2}$ where $K = 1, 2$.

The first order elastic moduli for hyperelastic incompressible solids are defined by the following equation:

$$L_{IJ\mathrm{KL}} = 2\,\frac{\partial T_{IJ}}{\partial C_{\mathrm{KL}}} \tag{28a}$$

where the tensor T_{KL} for hyperelastic incompressible solids is given by

$$T_{\mathrm{KL}} = \hat{T}_{\mathrm{KL}} - pC_{\mathrm{KL}}^{-1}. \tag{28b}$$

The second order tensor T_{IJ} may easily be computed using its eigenvectors $\underline{\mathrm{N}}$ and its eigenvalues [Eq. (21)]. With respect to the orthonormal basis i_k (k = 1, 2, 3) the components of the tensor T_{IJ} are given by

$$\hat{T}_{IJ} = N_{IK}\,\{\hat{T}\}_{\mathrm{KL}} N_{JL} \tag{29}$$

where $\{\mathbf{T_{KL}}\}$ is a diagonal matrix whose entries are the principal values of the stress tensor T [see Eq. (21)].

Assuming an Ogden type of strain energy function it can be proven[21] that the components of the material properties matrix with respect to the principal axes $\underline{\mathrm{N}}$ are

$$L_{\mathrm{KKKK}} = \sum_{i=1}^{m} (\alpha_i - 2)\mu_i \lambda_K^{\alpha_i - 4}; \quad K = 1, 2 \tag{30a}$$

$$L_{\mathrm{KLKL}} = \frac{(\hat{T}_K - \hat{T}_K)}{(\lambda_K^2 - \lambda_L^2)}\; K, L = 1, 2 \quad \lambda_K^2 \neq \lambda_K^2 \tag{30b}$$

$$L_{\mathrm{KLKL}} = \frac{L_{1111}}{2}, \quad K, L = 1, 2; \quad \lambda_K^2 = \lambda_L^2 \tag{30c}$$

with respect to the orthonormal basis i_k (k = 1–3) the material properties matrix is

$$C_{IJ\mathrm{KL}} = Q_{IM} Q_{\mathrm{JN}} \{L\}_{\mathrm{MNRS}} Q_{\mathrm{KR}} Q_{\mathrm{LS}}. \tag{31}$$

In this case, there is no condition to eliminate the pressure degree of freedom from the equilibrium equations. In order to eliminate the pressure degree of freedom the penalty function technique[14,16,29,30] is used. According to this method the pressure is replaced by:

$$p = \frac{1}{\varepsilon} G(I_3) \tag{32}$$

where ε is a small positive real number, the so-called penalty parameter and $G(I_3)$ is a continuous function of the third invariant of the G_{IJ} tensor. The function G is called the penalty function and satisfies the following properties:

$$G(1) = 0, \quad G(x) \neq 0 \quad \text{when } x \neq 1 \tag{33}$$

Along the course of this work we assume that the penalty function is a logarithmic function of the third invariant of the right Cauchy–Green strain tensor,

$$G(I_3) = -\frac{1}{2}\ln(I_3) \tag{34}$$

hence the pressure is given by

$$p = -\frac{1}{2\varepsilon}\ln(I_3). \tag{35}$$

The third invariant of the C_{IJ} tensor is given by:

$$I_3 = \det(C_{IJ}). \tag{36}$$

For incompressible nonlinear solids the motion is isochoric, which implies that I_3 must be equal to unity (that is, $I_3 \approx 1$). The principal values of the second Piola–Kirchhoff stress tensor are

$$T_j = \hat{T}_j - \frac{p_0 + p}{\lambda_j^2} \tag{37}$$

where the constant p_0 is introduced in order for the stress T_j to be equal to zero at the undeformed state. It can be easily seen from Eqs. (21), (38), and (40) that

$$p_0 = \sum_{n=1}^{m} \mu_n. \tag{38}$$

For two-dimensional problems the matrix $\mathbf{Q}$, which defines the orientation of the principal axes N_k with respect to the orthonormal basis i_k, is

$$[\mathbf{Q}] = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{39}$$

where the principal angle θ is given by: $\theta = \arccos(N_{11})$.

For a plane strain condition, it can be shown (see Appendix A) that the material properties matrix is

$$[\mathbf{C}] = [\hat{\mathbf{C}}] + [\mathbf{C}]_0 + [\mathbf{C}]_\varepsilon. \tag{40}$$

PRINCIPLE OF VIRTUAL WORK

As it was pointed out, the solid occupies the state B_0 at the undeformed state (i.e., at $t = 0$). At time t the configuration of the solid has been changed to B_1 and at $t = t + \Delta t$ the medium is at the configuration B_2. The *principle of virtual work* says that the work done on the deformed body from the external forces should be balanced from the internal forces, that is,

$$\int_{v_0}^{2} {}_0T_{\mathrm{KL}}\delta_0^2 E_{\mathrm{KL}}\, dV = {}^2R \tag{41}$$

where the backsubscript "0" indicates that the kinematic quantities are determined with respect to the reference configuration B_0 and the backsuperscript 2 denotes that the kinematic quantities are determined with respect to the configuration B_2. The ${}^2R$ denotes the incremental work performed by the external forces, that is body forces and surface fractions on the body. The external work is[27]

$$^2R = \int_{v_0} f_l \delta u_l \delta V + \int_{A_0} S_l \delta u_l\, dA \tag{42}$$

where f_1 and S_1 denote the body forces and the surface tractions, respectively. The δu_l defines an incremental displacement performed by the external forces on the body. Equations (41) and (42) define the differential form of the equilibrium equations of motion.

According to Bathe,[27] the *linearized* equations of variation are

$$\int_{V_0} {}_0^1T_{\mathrm{KL}}\delta(\Delta\eta_{\mathrm{KL}})\, dV + \int_{V_0} \delta(\Delta e_{\mathrm{KL}})C_{\mathrm{KLMN}}(\Delta e_{\mathrm{MN}})\, dV = {}^2R - \int_{V_0} {}_0^1T_{\mathrm{KL}}(\Delta e_{\mathrm{KL}})\, dV \tag{43}$$

where C_{KLMN} defines the material properties matrix and the quantities $\Delta\eta_{KL}$ and Δe_{KL} define the linear and nonlinear incremental parts of the strain tensor $\mathbf{E}_{KL}$.

FE DISCRETIZATION OF EQUILIBRIUM EQUATIONS

Following the traditional *isoparametric* FE discretization of the equilibrium equation,[3,27,35,36] Eq. (43), for two-dimensional plane strain problems, yields the following discretized FE equations in the form:

$$\left([K_0] + \frac{1}{\varepsilon}[K_\varepsilon]\right)\{\Delta U\} = \{R\} - \left(\{F\}_0 + \frac{1}{\varepsilon}\{F\}_\varepsilon\right) \quad (44)$$

where

$$[K_0] = \int_{V_0} [B_L]^t([\hat{C}] + [C]_0)[B_L]\, dV$$

$$+ \int_{V_0} [\mathrm{B}_{\mathrm{NL}}]^t([T_0] + [\hat{T}])[B_{nl}]\, dv \quad (45a)$$

$$[K_\varepsilon] = {}_{V_0} [B_L]^t[C_\varepsilon][B_L]\, dV \quad (45b)$$

$$\{F_0\} = \int_{V_0} [B_L]^t(\{T_0\} + \{\hat{T}\})\, dV \quad (45c)$$

$$\{F\}_\varepsilon = \int_{V_0} [B_L]^t\{T\}_\varepsilon\, dV \quad (45d)$$

$$\{\hat{T}\} = [\mathbf{Q}]\mathrm{diag}\{\hat{T}\}[\mathbf{Q}]^t \quad (45e)$$

$$\{T\}_0 = [\mathbf{Q}]\mathrm{diag}\left\{-\frac{p_0}{\lambda_1^2}\right\}[\mathbf{Q}]^t \quad (45f)$$

$$\{T\}_\varepsilon = [\mathbf{Q}]\mathrm{diag}\left\{\frac{-p}{\lambda_2^2}\right\}[Q]^t\, dV \quad (45g)$$

the matrix $[\mathbf{Q}]$ defines the rotation matrix given in Eq. (39), and the pressure terms p_0, p are defined in Eqs. (35) and (38).

NUMERICAL EXPERIMENTS

Consider the extension of a square elastomer sheet with a circular hole at the center of the sample. The geometry of the testing material and the FE mesh are shown in Figure 1. The thickness of the specimen was assumed to be infinitely long and a uniform pressure was applied along the two opposite edges of the sample in the x-direction. Because of the symmetry only one quarter of the sample was analyzed. The final extension of node A was $\lambda = 6.045$. The material of the testing solid was assumed to be: Mooney/Rivlin of material parameters $C_1 = 28.4$ psi, $C_2 = 1.42$ psi[35,37]; Ogden with material parameters $\alpha_1 = 1.3$, $\alpha_2 = 5$, $\alpha_3 = -2$, $\mu_1 = 89.446$, $\mu_2 = 0.1704$, $\mu_3 = -1.42$ psi[21]; Ogden with one term $\alpha_1 = 1.3$ and $\mu_1 = 8.875$ psi.[21]

Because the hole is small with respect to the dimensions of the rectangular block, the analytical solution of this problem can be readily obtained assuming that the block is substained as a pure shear deformation. The analytical solution of the block problem without the small infinitely long cylinder along the axis is shown in Figure 2 with solid lines. The solution was obtained for both Mooney–Rivlin and Ogden's type of material. The FE solution based on the Mooney–Rivlin and the Ogden's strain energy function is shown in Figure 2. Obviously, the FE

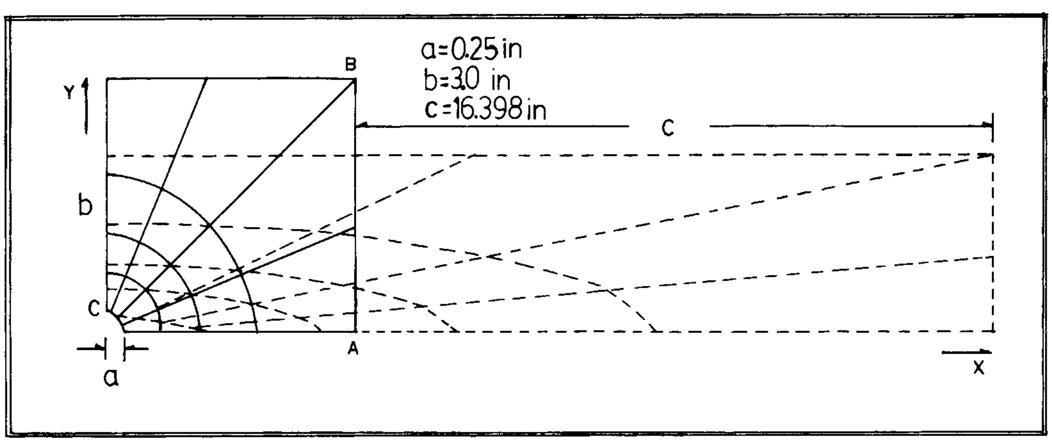

Figure 1 Rubber sheet in extension with a circular hole of radius a, at the center.

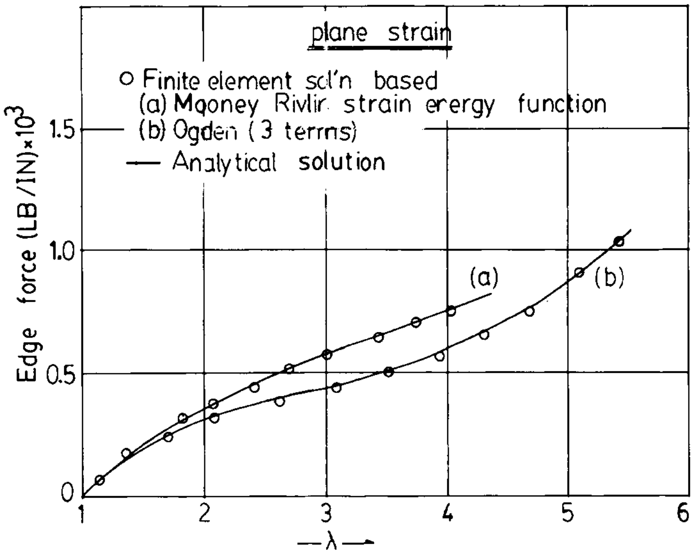

Figure 2 Edge force per unit thickness versus the extension ratio at point A.

solution based on the principal axes formulation yields accurate numerical results in the case of plane strain conditions.

As a second plane strain example, we consider the case of a finite deformation of an infinitely long thick-walled cylinder subjected to internal pressure p_i. For a Mooney–Rivlin material the analytical solution of this problem is given in Green and Zerna,[38] and for an Ogden material the solution is shown in Appendix B.

The internal pressure as a function of the extension ratio of the inner surface is shown in Figure 3 for different values of the n-parameter. The n-parameter defines the ratio of the inner to the outer radii of the cylinder. Obviously, there is a critical value of the n-parameter (say n_{cr}), such that for any $n \geq n_{cr}$ no limit points exist on the p_i vs. λ_a curve. In our FE model, a value of 0.25 was assigned for the n-parameter.

Figure 4 shows the FE mesh and the dimensions of the infinitely long thick-walled cylinder. The deformed shape of the cylinder for 50 and 300% deformation is depicted in the same figure. Due to the symmetry only one-quarter of the cylinder was analyzed. The FE mesh consisted of 80 elements with eight nodes in an element. The total of nodal points was 277, and the number of free degrees of freedom was 235. The solid line in Figure 5 defines the internal pressure p_i as a function of the λ_a (i.e., the extension of the internal surface) according to Eq. (B.1) with $m = 3$. The FE solution (shown with open circles) based on the Ogden's strain energy function with three terms (the material parameters are the same as in the previous example) agrees perfectly well with the analytical solution given by Eq. (B.1). The FE solution (shown with solid squares) based on the Ogden's strain energy function with one term (the same material parameters as in the previous

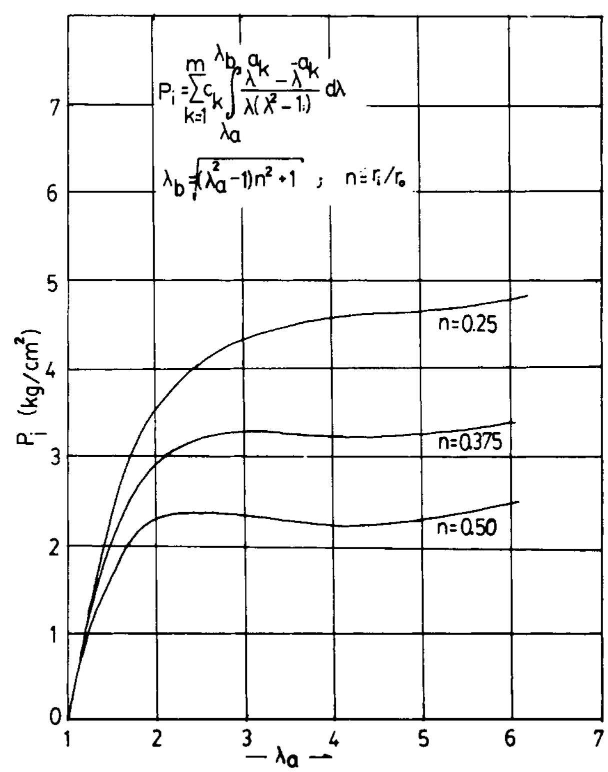

Figure 3 Internal pressure vs. the extension of the inner surface.

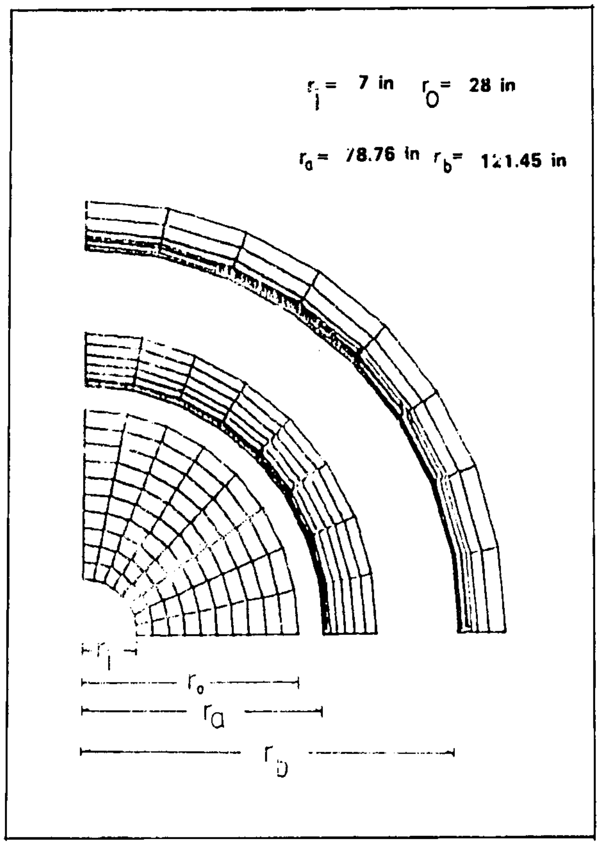

Figure 4 Finite element mesh of an infinitely long thicked-walled cylinder subjected to internal pressure. Deformed shape at 50 and 300% deformation.

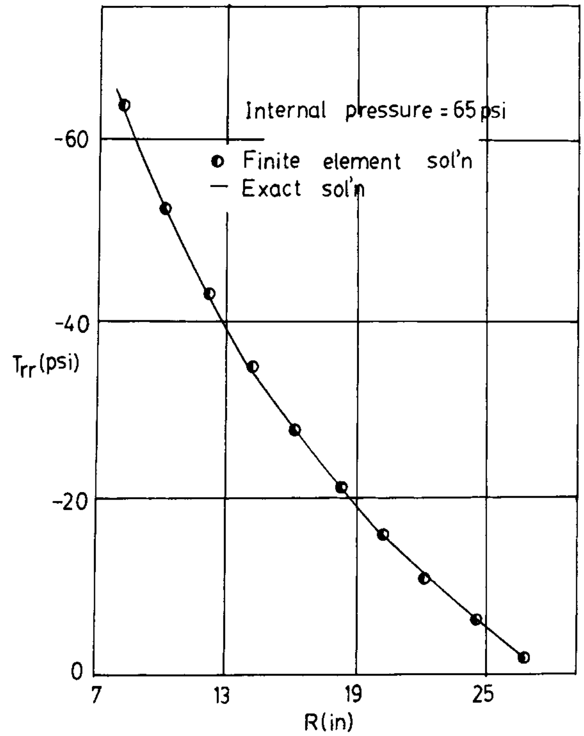

Figure 6 Radial stress vs. undeformed radial distance, *R* (cm).

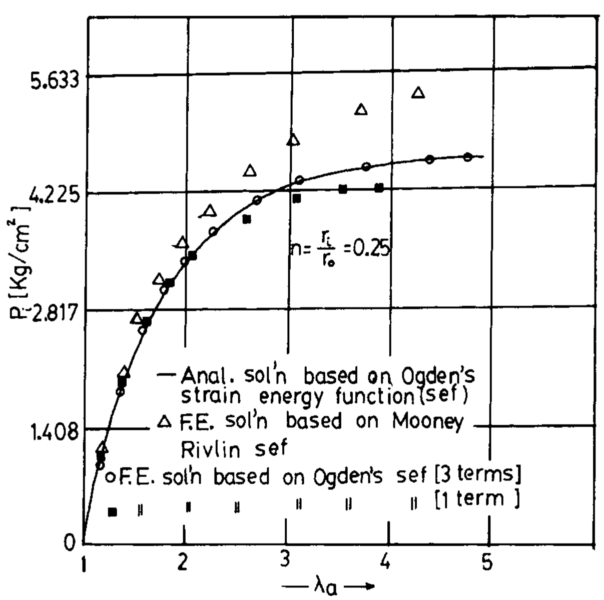

Figure 5 Internal pressure vs. extension of the inner surface using the Mooney–Rivlin and Ogden's strain energy function.

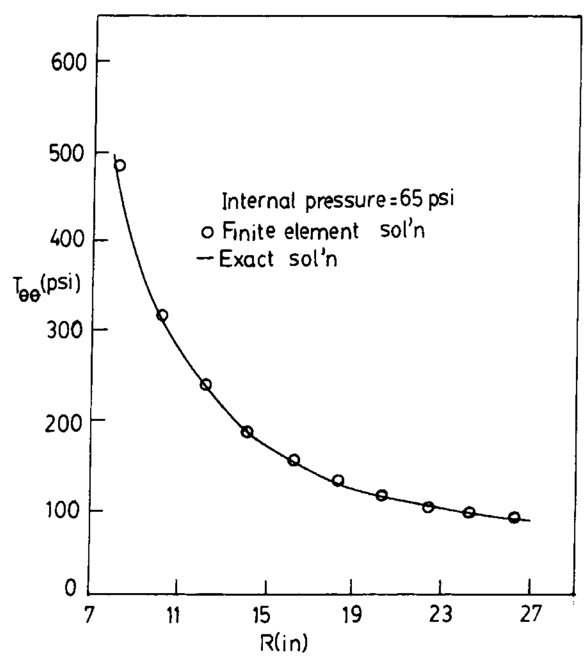

Figure 7 Circumferential stress distribution vs. undeformed radial distance.

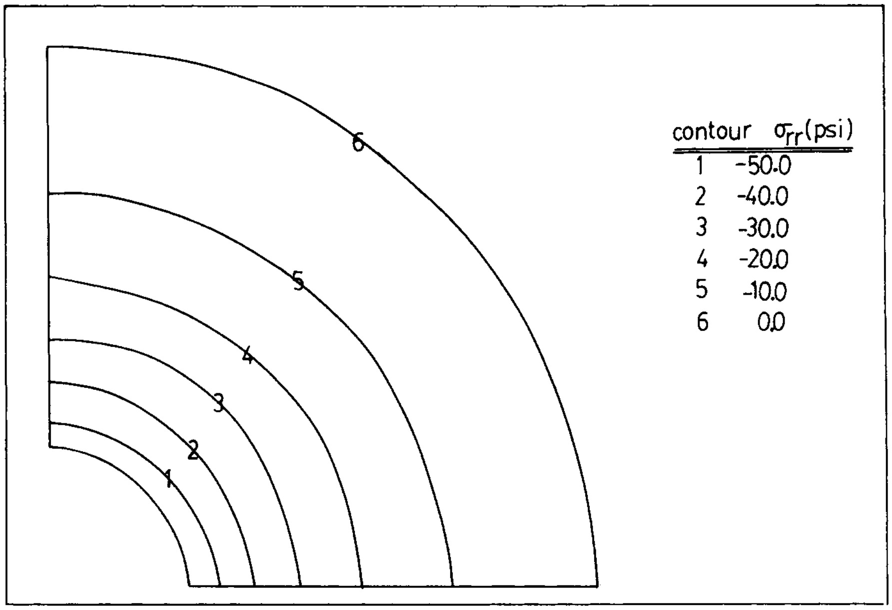

Figure 8 Contour map of radial stress field.

example) yields a good agreement with the analytical solution up to extension $\lambda_a = 2.2$ and after that it falls below the analytical curve. The FE solution (shown with open triangles) based on the Mooney/Rivlin strain energy function (with constants C_1, C_2 as in the previous case) follows the analytical solution up to $\lambda_\alpha = 1.8$. After that it falls above the analytical curve. Therefore, the FE solution based on three terms in the Ogden's strain energy function leads to the best agreement to the analytical solution of the present problem. A value of 0.0001 was assigned for the penalty parameter ε. In order to obtain a correct solution the number of load steps for the solution of the linearized equations was 50, and the number of equilibrium iterations per load step was less than five.

Figure 6 shows the radial stress distribution T_{rr} as a function of the radial distance R ($R_i \leq R \leq R_0$). The open circles represent the FE solution based on the Ogden's strain energy function with three terms. All the points closely follow the analytical solution of the stress T_{rr} given via Eqs. (B.3) and (B.5). The

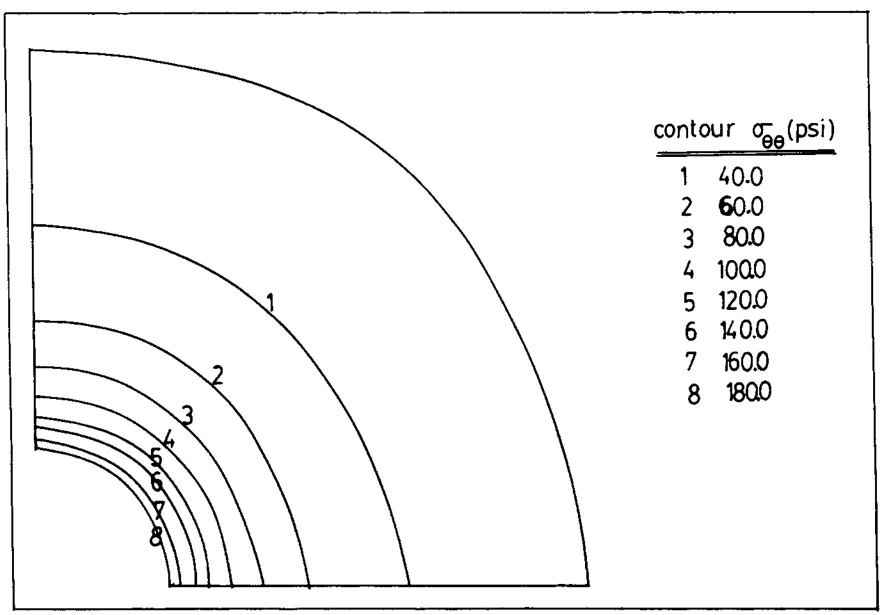

Figure 9 Contour map of the circumferential stress field.

internal pressure was 65 psi. Figure 7 shows the $T_{\theta\theta}$ component of the stress field as a function of R. Similar plots were reported by Ogden,[35] but the material was assumed to follow the Mooney–Rivlin strain energy function and the deformation was much lower than that reported in this work.

The contour maps of the radial and the circumferential stress distribution are depicted in Figures 8 and 9.

CONCLUSIONS

It has been shown that the proposed finite algorithm based on the penalty function formulation and the general measure of strain yields satisfactory numerical results for the solution of elastic incompressible solids subjected to large deformations. This model may be easily implemented into the existing displacement FE codes, for instance NONSAP, COSMOS7, ANSYS, ADINA, TEXLESP, etc.

APPENDIX A

Equation (40) can be proven as follows. For plane strain conditions the second Piola–Kirchhoff stress tensor is given by:

$$T_{\mathrm{KL}} = \hat{T}_{\mathrm{KL}} - (p_0 + p)C^{-1}_{\mathrm{KL}} \tag{A.1}$$

where the T_{KL} is given by (29) and the pressure terms p and p_0 via Eqs. (35) and (38), respectively. Taking the material derivatives of both sides of Eq. (A.1) we obtain,

$$T^{0}_{\mathrm{KL}} = 2[\hat{C}]_{\mathrm{KLMN}}C^{0}_{\mathrm{MN}} - 2\frac{\partial p}{\partial C_{\mathrm{MN}}}C^{0}_{\mathrm{MN}}C^{-1}_{\mathrm{KL}} + 2(p_0 + p)C^{-1}_{\mathrm{KM}}C^{0}_{\mathrm{MN}}C^{-1}_{\mathrm{NL}}. \tag{A.2}$$

Because,

$$T^{0}_{\mathrm{KL}} = 2[\mathbf{C}_{\mathrm{KLMN}}]C^{0}_{\mathrm{MN}} \tag{A.3}$$

Eq. (A.2) yields the overall stiffness matrix, that is

$$[\mathbf{C}] = [\hat{\mathbf{C}}] + [\mathbf{C}]_0 + [\mathbf{C}]_\varepsilon \tag{A.4}$$

which is identically equal to Eq. (40). The matrix $[\mathbf{C}]$ is given by Eq. (31) and the matrix $[\mathbf{C}]_0$ is defined by

$$[\mathbf{C}]_0 = p_0C^{-1}_{\mathrm{KM}}C^{-1}_{\mathrm{NL}}. \tag{A.5}$$

The matrix $[\mathbf{C}]_\varepsilon$ is defined by

$$[\mathbf{C}]_\varepsilon = pC^{-1}_{\mathrm{KM}}C^{-1}_{\mathrm{NL}} - \frac{\partial p}{\partial C_{\mathrm{MN}}}C^{-1}_{\mathrm{KL}}. \tag{A.6}$$

Analytically, the matrix $[\mathbf{C}]_0$ is written as follows:

$$[\mathbf{C}]_0 = \begin{bmatrix} 2p_0C_{22}^2 & 2p_0C_{12}^2 & -2p_0C_{12}C_{22} \\ & 2p_0C_{11}^2 & -2p_0C_{11}C_{12} \\ \text{SYMMETRIC} & & p_0(C_{11}C_{22} + C_{12}^2) \end{bmatrix} \tag{A.7}$$

The matrix $[\mathbf{C}]_\varepsilon$ may be written as follows:

$$[\mathbf{C}]_\varepsilon = \begin{bmatrix} (2p + A)C_{22}^2C_{33}^2 & (2p + A)C_{12}^2C_{33}^2 & -(2p + A)C_{12}C_{22}C_{33}^2 \\ & (2p + A)C_{11}^2C_{33}^2 & -(2p + A)C_{11}C_{12}C_{33}^2 \\ \text{SYMMETRIC} & & (2p + A)C_{12}^2C_{33}^2 + 2pC_{11}C_{22}C_{33}^2 \end{bmatrix} \tag{A.8}$$

where the pressure p is given via Eq. (35) and the constant A is equal to $1/\varepsilon$.

APPENDIX B

Let R_i and R_0 being the inner and outer radii of the cylinder. Following Ogden[21] it can be easily proven that the internal pressure as a function of the extension of the inner surface of the cylinder is

$$p_i = \sum_{k=1}^{m} \mu_k \int_{\lambda_a}^{\lambda_b} \frac{\lambda_k^\alpha - \lambda_k^\alpha}{\lambda(\lambda^2 - 1)}\, d\lambda \tag{B.1}$$

where the following relation holds between the extensions λ_a of the inner surface and the extension λ_b of the outer surface of the cylinder:

$$\lambda_b = ((\lambda_a^2 - 1)n^2 + 1)^{1/2}. \tag{B.2}$$

In the above formula n defines the ratio of the inner to the outer radius of the cylinder, that is, $n = R_i/R_0$.

The radial and circumferential stress field within the cylinder is given by[38]:

$$T_{rr} = -L(r) \tag{B.3}$$

$$T_{\theta\theta} = -L(r) + \frac{2R^2 b + b^2}{R^2(R^2 + b)} \Phi(r) \qquad \text{(B.4)}$$

where the functions $L(r)$ and $\Phi(r)$ are given by:

$$L(r) = -\sum_{k=1}^{m} \mu_k \int_{r_0=(R_0^2+b)^{1/2}}^{r=(R^2+b)^{1/2}} \left\{\left(\frac{r}{R}\right)^{\alpha_k} - \left(\frac{r}{R}\right)^{-\alpha_k}\right\} \frac{dr}{r} \qquad \text{(B.5)}$$

and

$$\Phi(r) = \sum_{k=1}^{m} \mu_k\left(\left(\frac{r}{R}\right)^{\alpha_k} - \left(\frac{R}{r}\right)^{\alpha_k}\right) \frac{R^2(R^2 + b)}{b(2R^2 + b)}. \qquad \text{(B.6)}$$

The constant b is given by

$$b = r_i^2 - R_i^2. \qquad \text{(B.7)}$$

Also the variable r is connected to the radial distance between the inner and the outer radii through the following equation: $r^2 = R^2 + b$.

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